Proving Equivalence

1. Prove that \( P \implies (Q \implies (R \implies S)) \) is equivalent to \( (P \land Q \land R) \implies S \).
   
   **Sample solution:**
   
   \[
   P \implies (Q \implies (R \implies S)) \iff \neg P \lor (\neg Q \lor (\neg R \lor S)) \quad \text{[implication rule]}
   \]
   
   \[
   \iff (\neg P \lor \neg Q \lor \neg R) \lor S \quad \text{[associativity of \lor]}
   \]
   
   \[
   \iff (\neg P \land \neg Q \land \neg R) \lor S \quad \text{[DeMorgan’s Law]}
   \]
   
   \[
   \iff (P \land Q \land R) \implies S \quad \text{[implication rule].}
   \]

2. Prove that \( ((P \implies Q) \implies R) \implies S \) is equivalent to \( (\neg P \land \neg R) \lor (Q \land \neg R) \lor S \).
   
   **Sample solution:**
   
   \[
   ((P \implies Q) \implies R) \implies S \iff (\neg (\neg P \lor Q) \lor R) \lor S \quad \text{[implication rule]}
   \]
   
   \[
   \iff ((\neg P \lor Q) \land \neg R) \lor S \quad \text{[DeMorgan’s Law]}
   \]
   
   \[
   \iff (\neg P \land \neg R) \lor (Q \land \neg R) \lor S \quad \text{[distributivity of \land]}
   \]

Negation

1. Every dog has its day, or perhaps its cat.
   
   **Sample solution:** Some dog has neither its day nor its cat.

2. \( \forall x \in X, \exists y \in Y, x > y \land y > x \)
   
   **Sample solution:** \( \exists x \in X, \forall y \in Y, x \leq y \lor y \leq x \)

Guarantees

Consider the statement:

(S1) A and B are both guarantees that C is true.

1. \( (A \implies C) \land (B \implies C) \) or \( (A \lor B) \implies C \)
2. “Being rich and being beautiful are both guarantees that one is hated.”

3. Suppose (S1) is true and A is false. What, if anything, can be determined about B and C? Briefly justify.
   Nothing. It tells us nothing about C, and A is unrelated to B.

4. Suppose (S1) is true and C is false. What, if anything, can be determined about A and B? Briefly justify.
   A is false and B is false. This comes from the contrapositive(s) of the implication(s), which must be true.