Work on these exercises before the tutorial. You don’t have to come up with complete solutions before the tutorial, but you should be prepared to discuss them with your TA.

1. Consider the statements below:

\[
\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n) \\
\exists n \in \mathbb{N}, P(n) \Rightarrow Q(n) \\
\forall n \in \mathbb{N}, Q(n) \Rightarrow P(n) \\
\exists n \in \mathbb{N}, Q(n) \Rightarrow P(n)
\]

Evaluate each of the four sentences above using each of the four definitions of predicates \(P\) and \(Q\) below. Briefly explain your evaluation of each of the sixteen cases.

(a) \(P(n) : n < 0\) \(Q(n) : n^2 < 0\)

**Solution:** \(P(n)\) and \(Q(n)\) are false for every natural number \(n\), so both the implication and its converse are vacuously true for every \(n \in \mathbb{N}\). So, all four sentences evaluate to true for this \(P\) and \(Q\).

(b) \(P(n) : n < 0\) \(Q(n) : n^2 > 17\)

**Solution:** The antecedent \(P(n)\) is false for every natural number \(n\), so the first two sentences are true for this \(P\) and \(Q\). \(n = 5\) is a counterexample to the third sentence, so it is false for this \(P\) and \(Q\). \(n = 2\) makes \(Q(n)\) false, and hence provides an example that the fourth sentence is vacuously true for this \(P\) and \(Q\).

(c) \(P(n) : n > 3\) \(Q(n) : n^2 < 0\)

**Solution:** \(n^2 \neq 0\) for all natural numbers \(n\), so any implication with \(Q(n)\) as the antecedent is vacuously true — the third and four sentences are true for this \(P\) and \(Q\). \(n = 2\) provides an example where \(P(n)\) is false, and hence making the second sentence true. \(n = 4\) is a counterexample making the first sentence false.

(d) \(P(n) : n > 3\) \(Q(n) : n^2 > 17\)

**Solution:** \(n = 4\) is a counterexample making the first sentence false for this \(P\) and \(Q\). \(n = 2\) is an example making the implication vacuously true in the second sentence. Since \(\sqrt{17} > 4\), the converse is true for every natural number, the third sentence is true for this \(P\) and \(Q\), and the particular case \(n = 5\) shows the fourth sentence is true for this \(P\) and \(Q\) (of course, any other natural number would do).

2. Consider the statement:

**S1:** For all students, missing an assignment or missing a quiz guarantees not getting a 100% in CSC165.
(a) Write $S_1$ in logical notation. The logical statement must be well-formed. Define all sets and predicate symbols that you use in the logical statement.

**Solution:** Let $S$ denotes the set of all students, $Q$ denotes the set of all quizzes, $A$ denotes the set of all assignments, $M(x, y)$ denotes $x$ missed $y$, and $F$ denotes $x$ gets full (100%) mark.

$$\forall x \in S, (\exists a \in A, M(x, a)) \lor (\exists q \in Q, M(x, q)) \Rightarrow \neg F(x).$$

**Alternative solution:** Let $S$ denotes the set of all students, $MQ(x)$ denotes $x$ missed a quiz, $MA(x)$ denotes $x$ missed an assignment, and $F$ denotes $x$ gets full (100%) mark.

$$\forall x \in S, MA(x) \lor MQ(x) \Rightarrow \neg F(x).$$

(b) Suppose $S_1$ is true and a student didn’t miss any assignments. What, if anything, can be determined about the student missing quizzes and getting 100% in CSC165? Briefly justify your answer.

**Solution:** Suppose $A$ denotes the student missed an assignment, $B$ denotes the student missed a quiz, and $C$ denotes the student didn’t get full mark.

We know that $S_1$ is true, so we have $A \lor B \Rightarrow C$.

We also know that $A$ is false. So, three cases are possible: (1) $B$ and $C$ are false. (2) $B$ is false and $C$ is true. (1) $B$ and $C$ are true.

So knowing that $A$ is false tells us nothing.

(c) Suppose $S_1$ is true and a student got 100% in CSC165. What, if anything, can be determined about the student missing assignments and quizzes? Briefly justify your answer.

**Solution:** We know that $C$ is false. So to keep $S_1$ true, the antecedent of the implication must be false, which means that $A$ and $B$ are false.

3. Translate the following sentences into logical notation. The logical sentences must be well-formed. Define all sets and predicate symbols that you use in the logical sentences.

(a) If some NP-complete problem can be solved efficiently, then every NP-complete problem can be solved efficiently.

**Solution:** Let $NPC$ denotes the set of all NP-complete problems and $SE(x)$ denotes that $x$ can be solved efficiently.

$$(\exists x \in NPC, SE(x)) \Rightarrow (\forall x \in NPC, SE(x)).$$

(b) Some courses have exactly one prerequisite course.

**Solution:** Let $C$ denotes the set of all courses and $P(y, x)$ denotes that $y$ is a prerequisite for $x$.

$$\exists x \in C, \exists y \in C, P(y, x) \land (\forall z \in C, P(z, x) \Rightarrow (z = y)).$$

(c) Some courses have the same prerequisite courses.

$$\exists x \in C, \exists y \in C, \forall z \in C, P(z, x) \leftrightarrow P(z, y)$$

(d) For every person, being a student is necessary for enrollment.

**Solution:** Let $P$ denotes the set of all persons, $E(x)$ denotes that $x$ is enrolled and $S(x)$ denotes $x$ is a student.

$$\forall x \in P, E(x) \Rightarrow S(x).$$

4. Give the contrapositive and converse of Question 3(d) in English.

**Contrapositive:** For every person, if the person is not a student, then s/he is not enrolled.

**Converse:** For every person, if the person is a student, then s/he is enrolled.

**Alternative #1:** For every person, being enrolled is necessary for being a student.

**Alternative #2:** For every person, being a student is sufficient for enrollment.