Do not turn this page until you have received the signal to start. (In the meantime, please fill out the identification section above, and read the instructions below.)
IMPORTANT: For all questions, you must use the proof structures and format of this course. Otherwise, you won’t get full mark even if your answer is correct.

**Question 1.** [13 marks]

Use the proof structure from this course to prove or disprove the following claims.

Note: You will receive part marks just for correctly writing the proof structure (aka the proof outline).

**Part (a) [6 marks]** $S_1$: For all integers $n$, $7n + 3$ is divisible by 7.

**Solution:** The claim is false, so I disprove it.

Here’s the translation of the claim:

$$\forall n \in \mathbb{Z}, 7 \mid (7n + 3).$$

To disprove it, I must prove the negation of the claim:

$$\exists n \in \mathbb{Z}, 7 \nmid (7n + 3).$$

Let $n = 0$. Then $n \in \mathbb{Z}$. # since $0 \in \mathbb{Z}$
Then $7n + 3 = 7 \times 0 + 3 = 3$. # substitute $n$ by 0
Then $7 \nmid 7n + 3$. # $7$ does not divide 3
Then $\exists n \in \mathbb{Z}, 7 \nmid (7n + 3)$. # introduce $\exists$

**Part (b) [7 marks]** $S_2$: For all integers if $n$ is even, then $7n + 4$ is even.

**Solution:** The claim is true.

Here’s the translation of the claim:

$$\forall n \in \mathbb{Z}, \text{Even}(n) \Rightarrow \text{Even}(7n + 4).$$

Assume $n \in \mathbb{Z}$. # $n$ is a typical integer
Assume $n$ is even. # antecedent
Then exists $k_0 \in \mathbb{Z}$ such that $n = 2k_0$. # definition of even numbers
Then $7n + 4 = 14k_0 + 4$. # substitute $n$ by $2k_0$ and algebra
Then exists $k_1 \in \mathbb{Z}$ such that $7n + 4 = 2k_1$. # $k_1 = 7k_0 + 2$, and $k_1 \in \mathbb{Z}$
Then $7n + 4$ is even. # definition of even numbers
Then $\text{Even}(n) \Rightarrow \text{Even}(7n + 4)$. # contrapositive is equivalent to the implication
Then $\forall n \in \mathbb{Z}, \text{Even}(n) \Rightarrow \text{Even}(7n + 4)$. # introduce $\forall$
Question 2. [12 marks]

Use the proof structure from this course to prove $S_3$.

Note: You will receive part marks just for correctly writing the proof structure (aka the proof outline).

$S_3$: For all integers $n$, if $n^2 + 5$ is odd, then $n$ is even.

Solution: Here's the translation of the claim:

$$\forall n \in \mathbb{Z}, \text{Odd}(n^2 + 5) \Rightarrow \text{Even}(n).$$

Assume $n \in \mathbb{Z}$. \# $n$ is a typical integer

Assume $n$ is odd. \# antecedent of the contrapositive

Then exists $k_0 \in \mathbb{Z}$ such that $n = 2k_0 + 1$. \# definition of odd numbers

Then $n^2 + 5 = (2k_0 + 1)(2k_0 + 1) + 5 = 4k_0^2 + 4k_0 + 6$. \# substitute $n$ by $2k_0 + 1$ and algebra

Then exists $k_1 \in \mathbb{Z}$ such that $n^2 + 5 = 2k_1$. \# $k_1 = 2k_0^2 + 2k_0 + 3$, and $k_1 \in \mathbb{Z}$

Then $n^2 + 5$ is even. \# definition of even numbers

Then $\text{Odd}(n) \Rightarrow \text{Even}(n^2 + 5)$. \# introduce $\Rightarrow$

Then $\text{Odd}(n^2 + 5) \Rightarrow \text{Even}(n)$. \# contrapositive is equivalent to the implication

Then $\forall n \in \mathbb{Z}, \text{Odd}(n^2 + 5) \Rightarrow \text{Even}(n)$. \# introduce $\forall$
Question 3. [15 marks]

Use proof by contradiction to prove that there is no integer \( n \) such that \( (n \equiv 5 \mod 6) \) and \( (n \equiv 3 \mod 12) \).

Note 1: You will receive part marks just for correctly writing the proof structure (aka the proof outline).

Note 2: You must use the proof structure from this course.

Hint: Recall that for integers \( x, y, z \), the notation \( x \equiv y \mod z \) means "\( x - y \) is a multiple of \( z \)."

Solution: Here's the translation of the claim:

\[ \neg (\exists n \in \mathbb{Z}, (n \equiv 5 \mod 6) \land (n \equiv 3 \mod 12)) \]

We must assume the negation of the claim and then derive a contradiction.

Assume \( \exists n \in \mathbb{Z}, (n \equiv 5 \mod 6) \land (n \equiv 3 \mod 12) \). \# to derive a contradiction

Then exists \( k_0 \in \mathbb{Z} \) such that \( n - 5 = 6k_0 \). \# definition \( \mod \)

Then \( n = 6k_0 + 5 \). \# add 5 to both sides of the above equality

Also exists \( k_1 \in \mathbb{Z} \) such that \( n - 3 = 12k_1 \). \# definition \( \mod \)

Then \( n = 12k_1 + 3 \). \# add 3 to both sides of the above equality

Then \( 12k_1 + 3 = 6k_0 + 5 \). \# both are equal to \( n \)

Then \( 12k_1 - 6k_0 = 2 \). \# subtract \( 6k_0 + 3 \) from both sides

Then \( 2k_1 - k_0 = 2/6 \). \# divide both sides by 6

Contradiction! \# \( 2k_1 - k_0 \in \mathbb{Z} \), but \( 2/6 \) is not an integer

Then \( \neg (\exists n \in \mathbb{Z}, (n \equiv 5 \mod 6) \land (n \equiv 3 \mod 12)) \). \# assuming the negation leads to a contradiction
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# 1: _____/13
# 2: _____/12
# 3: _____/15

TOTAL: _____/40