Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

1. Induction (summations). If marbles are arranged to form an equilateral triangle shape, with \( n \) marbles on each side, a total of \( \sum_{i=1}^{n} i \) marbles will be required.

In lecture, we proved that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \). For each \( n \in \mathbb{N} \), let \( T_n = \frac{n(n+1)}{2} \); these numbers are usually called the triangular numbers. Use induction to prove that

\[
\forall n \in \mathbb{N}, \quad \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}
\]
2. Induction (inequalities). Consider the statement:

For every positive real number $x$ and every natural number $n$, $(1 + x)^n \geq (1 + nx)$.

We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx$$

Notice that in this statement, there are two universally-quantified variables: $n$ and $x$. Prove the statement is true using the following approach:

(a) Use the standard proof structure to introduce $x$.
(b) When proving the $(\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx)$, do induction on $n$.

\[1\] Your predicate $P(n)$ that you want to prove will also contain the variable $x$ – that's okay, since when we do the induction proof, $x$ has already been defined.
3. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N}, n \leq 2^n$ using induction.

    (a) First, use trial and error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!

    \[
    \forall n \in \mathbb{N}, n \geq \quad \Rightarrow 30n \leq 2^n
    \]

    (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!