Learning objectives

By the end of this worksheet, you will:

- Know and apply various definitions for sets, strings, and common mathematical functions.
- Manipulate summation and product expressions.

1. **Set complement.** Consider the two sets $A$ and $U$ and suppose $A \subseteq U$. The complement of $A$ in $U$, denoted $A^c$, is the set of elements that are in $U$ but not $A$. Notice that this depends on the choice of both $U$ and $A$!

   (a) Let $U$ be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is $A^c$?

   (b) Write an expression for $A^c$ that uses the symbols $A$, $U$, and the set difference operator \.

   (c) Let $U$ represent the set of real numbers ($\mathbb{R}$), and consider the sets $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$ and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following, where the complement is taken with respect to $U$: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. Any observations?

2. **Set partitions.** A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is called a partition of a set $A$ if and only if (1) $A$ is the union of all of the $A_i$ and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any elements in common.

   (a) Let $\mathbb{Z}^+$ be the set of all positive integers, and let

   $\begin{align*}
   T_0 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\}, \\
   T_1 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 1, \text{ for some integer } k\}, \\
   T_2 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 2, \text{ for some integer } k\}, \\
   T_3 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}.
   \end{align*}$

   Write the first three elements of $T_0$, of $T_1$, of $T_2$, and of $T_3$.

   (b) Write down a partition of $\mathbb{Z}^+$ using $T_0$, $T_1$, $T_2$, and/or $T_3$. Why can’t you use all four sets?

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1 We say the $A_i$ are exhaustive.

2 We say the $A_i$ are mutually disjoint (or pairwise disjoint or nonoverlapping) if and only if no two sets $A_i$ and $A_j$ with distinct subscripts have any elements in common.
3. **Strings.** An **alphabet** $A$ is a set of symbols like $\{0, 1\}$ or $\{a, b, c\}$. A **string over alphabet** $A$ is a finite sequence of elements from $A$; the **length** of a string is simply the number of elements. Order matters in a string.

For example, 011 is a string over $\{0, 1\}$ of length three, and $abbbacc$ is a string over $\{a, b, c\}$ of length seven.

(a) Write down all strings over the alphabet $\{0, 1\}$ of length three (you should have eight in total).

(b) Let $S_1$ be the set of all strings over $\{a, b, c\}$ that have length two, and $S_2$ be the set of all strings over $\{a, b, c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \setminus S_2$.

(c) What do you notice about the relationship between $S_1$, $S_1 \cap S_2$, and $S_1 \setminus S_2$?

4. **The floor and ceiling functions.** Given any real number $x$, the **floor of** $x$, denoted $\lfloor x \rfloor$, is defined to be the largest integer that is less than or equal to $x$. Similarly, the **ceiling of** $x$, denoted $\lceil x \rceil$, is defined to be the smallest integer that is greater than or equal to $x$.

(a) What is the domain and range of the floor and ceiling functions?

(b) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of $x$: $x = \frac{25}{4}$, $x = 0.999$, and $x = -2.01$.

(c) Consider the following statement: For all real numbers $x$ and $y$, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Do you think this statement is True or False? Why?
5. Recall that the notation \( \sum_{i=j}^{k} f(i) \) gives us a short form for expressing the sum \( f(j) + f(j+1) + \cdots + f(k-1) + f(k) \), and that \( \prod_{i=j}^{k} f(i) \) gives us a short form for expressing the product \( f(j) \times f(j+1) \times \cdots \times f(k-1) \times f(k) \).

(a) Expand the following expressions to get the long sum/product they represent. Do not simplify.

\[
\begin{align*}
\sum_{k=1}^{3} (k + 1) & \quad \sum_{m=0}^{1} \frac{1}{2m} \\
\sum_{k=-1}^{2} (k^2 + 3) & \quad \sum_{j=0}^{4} \frac{(-1)^j}{j + 1} \\
\sum_{k=1}^{5} (2k) & \quad \prod_{i=2}^{4} \frac{i(i + 2)}{(i - 1)(i + 1)}
\end{align*}
\]

(b) Simplify each of the following expressions by using \( \sum \) or \( \prod \) notation.

\[
\begin{align*}
3 + 6 + 12 + 24 + 48 + 96 & \quad \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \frac{36}{729} \\
0 + 1 - 2 + 3 - 4 + 5 & \quad \left( \frac{1}{1+1} \right) \times \left( \frac{2}{2+1} \right) \times \left( \frac{3}{3+1} \right) \times \cdots \times \left( \frac{k}{k+1} \right) \\
\left( \frac{1 \cdot 2}{3 \cdot 4} \right) \times \left( \frac{2 \cdot 3}{4 \cdot 5} \right) \times \left( \frac{3 \cdot 4}{5 \cdot 6} \right)
\end{align*}
\]

6. It is not hard to prove manipulation results like the following that can be used to help us manipulate sums and products. If \( a_m, a_{m+1}, a_{m+2}, \ldots \) and \( b_m, b_{m+1}, b_{m+2}, \ldots \) are sequences of real numbers and \( c \) is any real number, then the following equations hold for any integer \( n \geq m \):

\[
\begin{align*}
\sum_{k=m}^{n} (a_k + b_k) & = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k \\
\sum_{k=m}^{n} c \cdot a_k & = c \cdot \sum_{k=m}^{n} a_k \\
\prod_{k=m}^{n} (a_k \cdot b_k) & = \left( \prod_{k=m}^{n} a_k \right) \left( \prod_{k=m}^{n} b_k \right)
\end{align*}
\]

Using these laws, rewrite each of the following as a single sum or product, but do not simplify your final sum/product. (You’ll learn late in the course how to do so.)

\[
\begin{align*}
3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k) & \quad \left( \prod_{k=1}^{n} \frac{k}{k+1} \right) \left( \prod_{k=1}^{n} \frac{k+1}{k+2} \right)
\end{align*}
\]