Learning objectives

By the end of this worksheet, you will:

- Know and apply various definitions for sets, strings, and common mathematical functions.
- Manipulate summation and product expressions.

1. **Set complement.** Consider the two sets $A$ and $U$ and suppose $A \subseteq U$. The complement of $A$ in $U$, denoted $A^c$, is the set of elements that are in $U$ but not $A$. Notice that this depends on the choice of both $U$ and $A$!

   (a) Let $U$ be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is $A^c$?

   **Solution**
   
   $A^c = \{1, 3, 4, 6\}$. 

   (b) Write an expression for $A^c$ that uses the symbols $A$, $U$, and the set difference operator \.

   **Solution**
   
   $A^c = U \setminus A$.

   (c) Let $U$ represent the set of real numbers ($\mathbb{R}$), and consider the sets $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$ and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following, where the complement is taken with respect to $U$: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. Any observations?

   **Solution**
   
   $A^c \cap B^c = \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\}$,
   
   $A^c \cup B^c = \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\}$,
   
   $(A \cap B)^c = \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\}$,
   
   $(A \cup B)^c = \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\}$.

   Note that: $A^c \cap B^c = (A \cup B)^c$ and $A^c \cup B^c = (A \cap B)^c$ (de Morgan’s laws for sets)

2. **Set partitions.** A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is called a partition of a set $A$ if and only if (1) $A$ is the union of all of the $A_i$ and (2) the sets $A_1, A_2, A_3, \ldots$ do not have any elements in common.

   (a) Let $\mathbb{Z}^+$ be the set of all positive integers, and let

   $T_0 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\}$,
   
   $T_1 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 1, \text{ for some integer } k\}$,
   
   $T_2 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k + 2, \text{ for some integer } k\}$,
   
   $T_3 = \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}$.

   Write the first three elements of $T_0$, of $T_1$, of $T_2$, and of $T_3$.

   **Solution**
   
   $T_0 = \{3, 6, 9, \ldots\}$, $T_1 = \{1, 4, 7, \ldots\}$, $T_2 = \{2, 5, 8, \ldots\}$, $T_3 = \{6, 12, 18, \ldots\}$.

   (b) Write down a partition of $\mathbb{Z}^+$ using $T_0$, $T_1$, $T_2$, and/or $T_3$. Why can’t you use all four sets?

   **Solution**
   
   The set $\{T_0, T_1, T_2\}$ is a partition of $\mathbb{Z}^+$, since, when any positive integer is divided by 3, the possible integer remainders are 0, 1, and 2. The sets $T_0$, $T_1$, $T_2$ list the numbers whose remainder when divided by 3 are 0, 1, or 2, respectively.

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1 We say the $A_i$ are **exhaustive**.
2 We say the $A_i$ are **mutually disjoint** (or pairwise disjoint or nonoverlapping) if and only if no two sets $A_i$ and $A_j$ with distinct subscripts have any elements in common.
Note that \( T_3 \subseteq T_0 \), so we can’t use both \( T_0 \) and \( T_3 \) in our partition (they have elements in common).
3. Strings. An alphabet $A$ is a set of symbols like \{0, 1\} or \{a, b, c\}. A string over alphabet $A$ is a finite sequence of elements from $A$; the length of a string is simply the number of elements. Order matters in a string.

For example, 011 is a string over \{0, 1\} of length three, and $abbacc$ is a string over \{a, b, c\} of length seven.

(a) Write down all strings over the alphabet \{0, 1\} of length three (you should have eight in total).

Solution
\{000, 001, 010, 011, 100, 101, 110, 111\}

(b) Let $S_1$ be the set of all strings over \{a, b, c\} that have length two, and $S_2$ be the set of all strings over \{a, b, c\} that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \setminus S_2$.

Solution
$S_1 \cap S_2 = \{aa, bb, cc\}$.
$S_1 \setminus S_2 = \{ab, ac, ba, bc, ca, cb\}$.

(c) What do you notice about the relationship between $S_1$, $S_1 \cap S_2$, and $S_1 \setminus S_2$?

Solution
Hint: look at $(S_1 \cap S_2) \cup (S_1 \setminus S_2)$.

4. The floor and ceiling functions. Given any real number $x$, the floor of $x$, denoted $\lfloor x \rfloor$, is defined to be the largest integer that is less than or equal to $x$. Similarly, the ceiling of $x$, denoted $\lceil x \rceil$, is defined to be the smallest integer that is greater than or equal to $x$.

(a) What is the domain and range of the floor and ceiling functions?

Solution
The domain is $\mathbb{R}$ and the range is $\mathbb{Z}$.

(b) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of $x$: $x = \frac{25}{4}$, $x = 0.999$, and $x = -2.01$.

Solution
$\lfloor \frac{25}{4} \rfloor = \lfloor 6.25 \rfloor = 6$, $\lceil \frac{25}{4} \rceil = \lceil 6.25 \rceil = 7$, $\lfloor 0.999 \rfloor = 0$, $\lceil 0.999 \rceil = 1$, $\lfloor -2.01 \rfloor = -3$, $\lceil -2.01 \rceil = -2$.

(c) Consider the following statement: For all real numbers $x$ and $y$, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Do you think this statement is True or False? Why?

Solution
The statement is False, since, for example, $\lfloor \frac{1}{2} + \frac{2}{3} \rfloor = \lfloor \frac{7}{6} \rfloor = 1$, while $\lfloor \frac{1}{2} \rfloor + \lfloor \frac{2}{3} \rfloor = 0 + 0 = 0$. 

5. Recall that the notation \( \sum_{i=j}^{k} f(i) \) gives us a short form for expressing the sum \( f(j) + f(j+1) + \cdots + f(k-1) + f(k) \), and that \( \prod_{i=j}^{k} f(i) \) gives us a short form for expressing the product \( f(j) \times f(j+1) \times \cdots \times f(k-1) \times f(k) \).

(a) Expand the following expressions to get the long sum/product they represent. Do not simplify.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{k=1}^{3} (k+1) )</td>
<td>( (1+1) + (2+1) + (3+1) )</td>
</tr>
<tr>
<td>( \sum_{k=-1}^{2} (k^2 + 3) )</td>
<td>( (1+3) + (0+3) + (1+3) + (4+3) )</td>
</tr>
<tr>
<td>( \sum_{k=1}^{5} (2k) )</td>
<td>( 2 + 4 + 6 + 8 + 10 )</td>
</tr>
<tr>
<td>( \sum_{m=0}^{1} \frac{1}{2^m} )</td>
<td>( \frac{1}{2^0} + \frac{1}{2^1} )</td>
</tr>
<tr>
<td>( \sum_{j=0}^{4} (-1)^{\frac{j}{j+1}} )</td>
<td>( 0 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} )</td>
</tr>
<tr>
<td>( \prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)} )</td>
<td>( \frac{2 \cdot 4 \cdot 3 \cdot 5 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 5} )</td>
</tr>
</tbody>
</table>

(b) Simplify each of the following expressions by using \( \sum \) or \( \prod \) notation.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Simplification</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 + 6 + 12 + 24 + 48 + 96 )</td>
<td>( \sum_{i=0}^{5} 3 \cdot 2^i )</td>
</tr>
<tr>
<td>( 0 + 1 - 2 + 3 - 4 + 5 )</td>
<td>( \sum_{j=0}^{5} (-1)^{\frac{j+1}{j+1}} \cdot j )</td>
</tr>
<tr>
<td>( \frac{1 \cdot 2}{3 \cdot 4} \times \frac{2 \cdot 3}{4 \cdot 5} \times \frac{3 \cdot 4}{5 \cdot 6} )</td>
<td>( \prod_{j=1}^{3} \frac{j \cdot (j+1)}{(j+2) \cdot (j+3)} )</td>
</tr>
</tbody>
</table>

6. It is not hard to prove manipulation results like the following that can be used to help us manipulate sums and products. If \( a_m, a_{m+1}, a_{m+2}, \ldots \) and \( b_m, b_{m+1}, b_{m+2}, \ldots \) are sequences of real numbers and \( c \) is any real number, then the following equations hold for any integer \( n \geq m \):

\[
\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k \\
\sum_{k=m}^{n} c \cdot a_k = c \cdot \sum_{k=m}^{n} a_k \\
\prod_{k=m}^{n} (a_k \cdot b_k) = \left( \prod_{k=m}^{n} a_k \right) \left( \prod_{k=m}^{n} b_k \right)
\]

Using these laws, rewrite each of the following as a single sum or product, but do not simplify your final sum/product. (You’ll learn late in the course how to do so.)
Solution

\[ 3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k) = 6 \cdot \left( \sum_{k=1}^{n} k \right) - 9 \cdot \left( \sum_{k=1}^{n} 1 \right) + 4 \cdot \left( \sum_{k=1}^{n} 1 \right) - 5 \cdot \left( \sum_{k=1}^{n} k \right) \]

\[ = \sum_{k=1}^{n} (k - 5) \]

\[ \left( \prod_{k=1}^{n} \frac{k}{k+1} \right) \left( \prod_{k=1}^{n} \frac{k+1}{k+2} \right) = \left( \prod_{k=1}^{n} \frac{k}{k+1} \cdot \frac{k+1}{k+2} \right) \]

\[ = \left( \prod_{k=1}^{n} \frac{k}{k+2} \right) \]