Week 11 graphs - Trees

Homework 4 out

Test 2 this week

- Binary/decimal representation
- Induction
- Big O, Ω, Θ (asymptotic) expressions
- Runtime analysis
Graphs are everywhere!

Example:
Definitions

A graph is a tuple of sets \((V, E)\) where

1. \(V\) is a set, where each element is called a vertex
2. \(E\) is a set of edges where an edge is a pair of vertices

\[ E = \{ (v_1, u_1), (v_2, u_2), (v_3, u_3), \ldots \} \]

where all vertex \(v\)’s and \(u\)’s are in \(V\)
- Order doesn't matter in an edge $(v, u)$ is actually $\{v, u\}$ (graphs are "undirected")
- $u_i \neq v_i$ (no "loops")
- at most one edge between 2 vertices
- Order doesn't matter in an edge \((v, u)\) is actually \([v, u]\) (graphs are "undirected")
- \(u \neq v\) (no "loops")
- at most one edge between two vertices

**Example**

\[
V = \{v_1, v_2, v_3, v_4, v_5\}
\]

\[
E = \{ (v_1, v_5), (v_2, v_5), (v_4, v_5), \\
(v_1, v_3), (v_2, v_3), (v_3, v_4) \}
\]
How many edges can there be in a graph with $n$ vertices?

**Theorem**

For all graphs $g = (V, E)$, $|E| \leq \frac{n(n-1)}{2}$, where $|V| = n$.

**Example** $|V| = 4$
Show: The \# of edges when \(|V| = n\) is
\[
\leq \frac{n(n-1)}{2}
\]

1. Order vertices \(v_1, \ldots, v_n\)
   - \# of edges out of \(v_1\) (containing \(v_1\)) is at most \((n-1)\)
   - \# of edges containing \(v_2\) but not \(v_1\) is at most \((n-2)\)

Total count is
\[
\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}
\]
How many edges can there be in a graph with n vertices?

**Theorem**

For all graphs $G = (V, E)$,

$$|E| \leq \frac{n(n-1)}{2}, \text{ where } |V| = n$$

**Proof**

- Each edge is a subset of exactly 2 vertices.
- So max # of edges is the number of subsets of $V$ of size 2.
- This number is $\frac{n(n-1)}{2} \in O(n^2)$.
Let \( g = (V, E) \) be a graph, and \( u, v \in V \)

We say \( u \) and \( v \) are neighbors or are adjacent when \( (u, v) \in E \).
Let $g = (V, E)$ be a graph, and $u, v \in V$. We say $u$ and $v$ are neighbors or are adjacent when $(u, v) \in E$.

A path between $u$ and $v$ is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ where:

1. $v_0 = u$
2. $v_k = v$
3. $\forall i \in \{0, 1, \ldots, k-1\}$, $v_i$ and $v_{i+1}$ are adjacent
4. All the vertices $v_0, v_1, \ldots, v_k$ are distinct
Let $g = (V, E)$ be a graph, and $u, v \in V$. We say $u$ and $v$ are neighbors or are adjacent when $(u, v) \in E$.

The length of a path is the number of edges.

Length is $k$.
Example

- $e \rightarrow e: v_0 = e$, length 0 path
- $e \rightarrow d: v_0 = e, v_1 = d$, length 1
- $b \rightarrow d: v_0 = b, v_1 = c, v_2 = d$, length 2
  - $b - a - c - d$, 3
  - $b - e - d$, 2

- Distance $(b,d)$: 2
- Distance $(b,b)$: 0
- Distance $(g,f)$: \infty

- A path can have length 0 (we allow $u = v$)
- We also allow $(u,v) \in E$ - length 1 paths

- Distance $(u_1,u_2) = 1$
- Distance $(u_1,u_4) = 2$
• There can be more than one path between u, v

• There can be paths of different lengths from u to v

• There can also be no paths from u to v

• The distance from u to v

  is the length of the shortest path from u to v
  (if no path exists, distance is \( \infty \))

• u and v are connected if there is a path between them
• A graph $G = (V, E)$ is **fully-connected** if $\forall u, v \in V$, $u$ and $v$ are connected.

**Example**

- Fully connected
- Not fully connected
\text{conn}(g, u, v): \text{true if } u \text{ and } v \text{ are connected in } g [\text{there is a path from } u \text{ to } v]

\[ \forall g \exists u, v, w \in V \quad \text{conn}(g, u, u) \]

\begin{align*}
(1) \quad & \text{conn}(g, u, v) \implies \text{conn}(g, v, u) \\
(2) \quad & \text{conn}(g, u, v) \land \text{conn}(g, v, w) \implies \text{conn}(g, u, w)
\end{align*}
path from u to w just concatenate the 2 paths

Harder situation (requiring a repair):
path from $u$ to $w$ just concatenate the 2 paths

Harder situation (requiring a repair):

New sequence: $a_0, a_1, a_2, b_5$
Proof of Transitivty

We want to prove
\[ \text{Conn}(g, u, v) \land \text{Conn}(g, v, w) \Rightarrow \text{Conn}(g, u, w) \]

Since \( \text{Conn}(g, u, v) \), there is a path \( u, a_1, a_2, \ldots, a_k, v \) of distinct vertices s.t. \( (u, a_1) \in E, (a_i, a_{i+1}) \in E, (a_k, v) \in E \)

ie. \( u \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots \rightarrow a_k \rightarrow v \)

and since \( \text{Conn}(g, v, w) \) there is a path \( b_1, b_2, \ldots, b_m \) of distinct vertices s.t. \( (v, b_1), (b_i, b_{i+1}), (b_m, w) \in E \)

ie. \( v \rightarrow b_1 \rightarrow b_2 \rightarrow \ldots \rightarrow b_m \rightarrow w \)

Case 1
If \( a_1, \ldots, a_k, b_1, \ldots, b_m \) are all distinct, then
then \( u-a_1-a_2-\ldots-a_k-v-b_1-b_2-\ldots-b_m-w \) is a path connecting \( u \) to \( w \)
Case 2 \( u, a_1, \ldots, a_k, v, b_1, \ldots, b_m, w \) not distinct

Pick smallest \( i \) such that \( q_i = b_j \) for some \( j \in 1 \ldots m \)

Then \( u - a_1 - a_2 - \ldots - a_i - b_{j+1} - b_{j+2} - \ldots - b_m - w \)

is a path connecting \( u \) to \( w \)

(and all vertices are distinct)
What is smallest # edges if \( G \) is connected?

**Theorem**

\[ \forall n \in \mathbb{N} \quad \forall G = (V, E) \quad (|V| = n \land G \text{ is connected}) \implies |E| \geq n - 1 \]

**High level of proof:**
- Trees are minimally connected graphs (they have no cycles)
- Any tree has exactly \( n-1 \) edges

\[ n = 7 \]
Defn. Let $G = (V, E)$ be a graph. A cycle is a sequence of vertices $v_0, v_1, \ldots, v_k$ where

1. $v_0 = v_k$
2. All other vertices are distinct
3. All $v_i, v_{i+1}$ are adjacent
4. $k \geq 3$

Ex. 

\[ \begin{align*} 
    v_0 &= v_3 \\
    v_1 &
\end{align*} \]
Main Cycle Theorem

(A graph is minimally connected $\iff$ it is connected and has no cycles)

$\forall G = (V, E)$ (g is connected and $G$ has a cycle) $\iff (\exists e \in E, G - e$ is connected)

$G$ is minimally connected means if you remove any edge, it is no longer connected.

The graph with edge $e$ removed.
Proof sketch (⇒)

Let $G = (V, E)$ and assume $G$ is connected and has a cycle.

Example:
Proof idea ($\Rightarrow$)

Let $G = (V, E)$ and assume it is connected and has a cycle. We want to show $\exists e \in E$ so that $G - e$ is still connected.

Can remove any edge in a cycle, the graph will still be connected! 

$2 - 4 - 5 - 9$
\begin{itemize}
\item \textbf{Suppose that}\n\item \textbf{G is connected}.
\item \textbf{Want to show that} for any edge \( e \) not in \( G \),
\item \( G' = G \) plus \( e \) must contain a cycle.
\end{itemize}

\textbf{Start with}

\[ G \]

\[ \exists \text{ path from } u \rightarrow v \text{ in } G. \]
Let $g$ be a connected graph. Let $e$ be a new edge not in $g$. We want to show $g' = g$ plus $e$ has a cycle.

Let $e = (v, v)$

Since $g$ is connected, there is a sequence of vertices $V_0, V_1, \ldots, V_k$ with $V_0 = v$, $V_k = v$, and all $V_i$, $V_{i+1}$ are adjacent.

Then the sequence $V_0, V_1, \ldots, V_k, V_0$ is a cycle in $g$ plus $e$ is a cycle.
Def'n (Trees)

Let $G = (V, E)$. We say that $G$ is a tree if $G$ is connected and has no cycles.

From Main Cycle Theorem, if $G$ is a tree, then $\forall e \in E \; g - e$ is not connected.
Summary so far:

We want to prove that for any graph $g$, if $g$ is connected, then $g$ has $\geq n-1$ edges.

1. **Main cycle theorem** says any $g$ that is connected and that has the **minimum number of edges** is a tree (has no cycles).

2. It is left to show (next class) that any tree has exactly $n-1$ edges.