1. Consider the definition of the floor function:

\[ D_1 : \forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, (y = \lfloor x \rfloor) \leftrightarrow (y \leq x) \land (\forall z \in \mathbb{Z}, (z \leq x) \Rightarrow (z \leq y)). \]

Use \( D_1 \) to prove \( \forall x \in \mathbb{R}, (\lfloor x \rfloor > x - 1) \).

**Proof:**

Assume \( x \in \mathbb{R} \). \# \( x \) is a typical element of \( \mathbb{R} \)

Then \( \lfloor x \rfloor, \lfloor x \rfloor + 1 \in \mathbb{Z} \). \# by definition of the floor function and \( \mathbb{Z} \) is closed under +

And \( \lfloor x \rfloor + 1 > \lfloor x \rfloor \). \# add \( \lfloor x \rfloor \) to \( 1 > 0 \)

Then \( \lfloor x \rfloor + 1 > x \). \# by contrapos. of the implication in \( D_1 \) which is \( \forall z \in \mathbb{Z}, (z > \lfloor x \rfloor) \Rightarrow (z > x) \)

Then \( \lfloor x \rfloor > x - 1 \). \# subtract 1 from both sides

Then \( \forall x \in \mathbb{R}, (\lfloor x \rfloor > x - 1) \). \# introduced \( \forall \)

2. \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor \).

**Solution:** To derive a contradiction, we assume the negation of the claim.

**Proof by contradiction:**

Assume \( \neg(\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor) \) \# to derive contradiction

Then \( \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, (x > y) \land (\lfloor x \rfloor < \lfloor y \rfloor) \) \# the negation

Let \( x_0 \in \mathbb{R}, y_0 \in \mathbb{R} \) be such that \( (x_0 > y_0) \land (\lfloor x_0 \rfloor < \lfloor y_0 \rfloor) \). \# instantiate \( \exists \)

Then \( \lfloor x_0 \rfloor < \lfloor y_0 \rfloor \) \# conjunction elimination

And \( \lfloor x_0 \rfloor \in \mathbb{Z}, \lfloor y_0 \rfloor \in \mathbb{Z} \) \# by definition of floor

Then \( \lfloor x_0 \rfloor + 1 \leq \lfloor y_0 \rfloor \) \# the smallest possible difference between two distinct integers is 1

Then \( \lfloor x_0 \rfloor + 1 \leq y_0 \) \# since \( \lfloor y_0 \rfloor \leq y_0 \) by definition of \( \lfloor y_0 \rfloor \)

Then \( \lfloor x_0 \rfloor + 1 < x_0 \) # \( y_0 < x_0 \) by the assumption and < is transitive

And \( \lfloor x_0 \rfloor + 1 \in \mathbb{Z} \) # 1, \( \lfloor x_0 \rfloor \in \mathbb{Z} \) and \( \mathbb{Z} \) is closed under +

Then \( \lfloor x_0 \rfloor + 1 \leq \lfloor x_0 \rfloor \) \# by definition of \( \lfloor x_0 \rfloor \) that \( \forall z \in \mathbb{Z}, z \leq x_0 \Rightarrow z \leq \lfloor x_0 \rfloor \)

Then \( 1 \leq 0 \) \# subtract \( \lfloor x_0 \rfloor \) from both sides, and contradiction with that \( 1 > 0 \)

Then \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y \Rightarrow \lfloor x \rfloor \geq \lfloor y \rfloor \) \# negation of assumption because of contradiction
3. For $x \in \mathbb{R}$, define $|x|$ by

$$|x| = \begin{cases} 
-x, & x < 0 \\
 0, & x \geq 0 
\end{cases}$$

Prove that $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$.

**Solution:** For each variable we must consider two cases: in one case the variable is $\geq 0$ and in the other case the variable is $< 0$. Since we have two variables we must have four cases in our proof.

**Proof:**

Assume $x, y \in \mathbb{R}$. # $x$ and $y$ are typical elements of $\mathbb{R}$

**Case 1:** $x < 0$ and $y < 0$.

Then $|x| = -x$ and $|y| = -y$. # definition of $|x|$ and $|y|

Then $|x||y| = (-x)(-y) = xy$. # since $(−1)^2 = 1$

And $xy > 0$. # the product of two negative numbers is positive

Then $xy = |xy|$. # definition of $|xy|$ when $xy \geq 0$ Then $|x||y| = |xy|$.

**Case 2:** $x < 0$ and $y \geq 0$.

Then $|x| = -x$ and $|y| = y$. # definition of $|x|$ and $|y|

Then $|x||y| = -xy$. # algebra

And $xy \leq 0$. # product of a negative and a non-negative number is either 0 or negative

**Case 2.1:** Assume $xy < 0$.

Then $|xy| = -xy$. # by the definition of $|xy|$

**Case 2.2:** Assume $xy = 0$.

Then $|xy| = 0 = -xy$. # product of any number and 0 is 0, and by the definition of $|xy|$

Then $|x||y| = |xy|$. # we showed that both are equal to $-xy$

**Case 3:** $x \geq 0$ and $y < 0$.

Then $|x| = x$ and $|y| = -y$. # definition of $|x|$ and $|y|

Then $|x||y| = -xy$. # algebra

And $xy \leq 0$. # product of a non-negative number with a negative number is non-positive

**Case 3.1:** Assume $xy < 0$.

Then $|xy| = -xy$. # by the definition of $|xy|$

**Case 3.2:** Assume $xy = 0$.

Then $|xy| = 0 = -xy$. # product of any number and 0 is 0, and by the definition of $|xy|$

Then $|x||y| = |xy|$. # true for both possible cases

**Case 4:** $x \geq 0$ and $y \geq 0$.

Then $|x||y| = xy$. # $|x| = x$ and $|y| = y$ by definition of $|x|$ and $|y|

And $xy \geq 0$. # the product of two non-negative numbers is non-negative

Then $|xy| = xy$. # definition of $|xy|$

Then $|x||y| = |xy|$. # both are equal to $xy$

Then $|x||y| = |xy|$. # true for all possible cases

Then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x||y| = |xy|$. # introduced $\forall$