Chapter 3

Formal Proofs

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Announcements

- **Term Test 1:**
  - Section **L0101**: Tuesday **Feb 03, 2:10-3:30** Location: **MP203**
  - Section **L0201**: Thursday **Feb 05, 2:10-3:30** Location: **MP103**

- You **must** write the quiz in the section that you are **enrolled** in, unless you have talked to the instructors and they allowed you to switch.

- **Content:** **Chapter 2.** Review lecture and course notes!

- **TA office Hours:**
  - **Monday,** Feb 02, 1-3pm, 4:30-6:30pm in **BA3201**
  - **Wednesday,** Feb 04, 12-2pm, 3:30-5:30pm in **BA3201**
Today’s Topics

- Direct Proof of the Existential
- Proof of Multiple Quantifiers, Implications, and Conjunctions
  - Example of Proving a Statement about a Sequence
  - Example of Disproving a Statement about a Sequence
Chapter 3

Formal Proofs

Direct Proof of the Existential
Direct Proof of the Existential

**General Form**

- **Prove:** \( \exists x \in D, P(x) \).
- **How to prove:**
  - Find **one** element in \( D \) that **satisfies** \( P \).

**Structure for the Direct Proof of Existential**

Let \( x = \ldots \)  # choose a particular element of the domain
Then \( x \in D \)  # this **may be obvious**, otherwise prove it

\[ \vdash \text{prove } P(x) \]
Then \( P(x) \)  # you’ve shown that \( x \) **satisfies** \( P \)
\( \exists x \in D, P(x) \)  # introduce existential
Direct Proof of the Existential

Exercise

- Prove $\exists x \in \mathbb{R}, x^2 + 2x + 1 = 0$.

  Let $x = -1$.  # choose a particular element that will work
  Then $x \in \mathbb{R}$.  # since $-1 \in \mathbb{R}$
  Then $x^2 + 2x + 1 = (-1)^2 + 2(-1)^2 + 1 = 1 - 2 + 1 = 0$.  # substitute $-1$ for $x$
  Then $\exists x \in \mathbb{R}, x^2 + 2x + 1 = 0$.  # we gave an example, so we introduce existential
Exercise

Prove $\exists x \in \mathbb{Z}, x = -x$.

Let $x = 0$. # choose a particular element that will work
Then $x \in \mathbb{Z}$. # since $0 \in \mathbb{Z}$
Then $x = 0 = -0 = -x$. # substitute 0 for $x$
Then $\exists x \in \mathbb{Z}, x = -x$. # we gave an example, so we introduce existential
Chapter 3

Formal Proofs

Proof of Multiple Quantifiers, Implications, and Conjunctions
Proving a Statement about a Sequence

• **A₁**: \( a_i = i^2, \ i \in \mathbb{N}. \)

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• **C₁**: \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \ a_j \leq i \implies j < i. \)

Verify if **C₁** is **True** for **A₁**.

• \( i = 0 \): \( a_0 = 0^2 \leq 0, \) but \( 0 \ngeq 0. \)
• \( i = 1 \): \( a_1 = 1^2 \leq 1, \) but \( 1 \ngeq 1. \)
• \( i = 2 \):
  • \( a_0 = 0^2 \leq 2, \) and \( 0 < 2. \)
  • \( a_1 = 1^2 \leq 2, \) and \( 1 < 2. \)
  • For all \( j \geq 2, \ a_j = j^2 \not\leq 2. \)
Proving a Statement about a Sequence

\[ A_1 : a_i = i^2, \, i \in \mathbb{N}. \]

\[ C_1 : \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i. \]

Prove that \( C_1 \) is True for \( A_1 \).
Reminder: Direct Proof of Universally Quantified Implication

**Structure of an Direct Proof**

- **Prove** $\forall x \in D, P(x) \Rightarrow Q(x)$
  - Assume $x \in D$.  $\# x$ is a typical element of $D$
    - Assume $P(x)$.  $\# x$ has property $P$, the antecedent
      - Then $Q(x)$.  $\#$ the **consequence**!
    - Then $P(x) \Rightarrow Q(x)$.  $\#$ assuming antecedent leads to consequent
  - Then $\forall x \in D, P(x) \Rightarrow Q(x)$.  $\# x$ was a typical element of $D$
Reminder: Indirect Proof of Universally Quantified Implication

Structure of an Indirect Proof

- Prove $\forall x \in D, P(x) \Rightarrow Q(x)$
  
  Assume $x \in D$.  # $x$ is a typical element of $D$
  Assume $\neg Q(x)$.  # negation of the consequent!
  
  :  
  Then $\neg P(x)$.  # negation of the antecedent!
  Then $\neg Q(x) \Rightarrow \neg P(x)$.  # assuming $\neg Q(x)$ leads to $\neg P(x)$
  Then $P(x) \Rightarrow Q(x)$.  # implication is equivalent to contrapositive
  Then $\forall x \in D, P(x) \implies Q(x)$.  # $x$ was a typical element of $D$
Proving a Statement about a Sequence

- **A₁**: \(a_i = i^2, i \in \mathbb{N}\).
- **C₁**: \(\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i\).

Prove that **C₁** is **True** for **A₁**.

Let \(i = 2\). Then \(i \in \mathbb{N}\). \# 2 \in \mathbb{N}

\[\vdash\]

Then \(\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i\). \# introduce existential
Proving a Statement about a Sequence

- **A₁**: \( a_i = i^2, i \in \mathbb{N} \).
- **C₁**: \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i \).

Prove that **C₁** is True for **A₁**.

Let \( i = 2 \). Then \( i \in \mathbb{N} \). \# \( 2 \in \mathbb{N} \)

Assume \( j \in \mathbb{N} \). \# typical element of \( \mathbb{N} \)

\[ \vdots \]

Then \( \forall j \in \mathbb{N}, a_j \leq i \implies j < i \). \# introduce universal

Then \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i \). \# introduce existential
Proving a Statement about a Sequence

- **A<sub>1</sub>**: \( a_i = i^2, i \in \mathbb{N} \).
- **C<sub>1</sub>**: \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i \).

Prove that **C<sub>1</sub>** is **True** for **A<sub>1</sub>**.

Let \( i = 2 \). Then \( i \in \mathbb{N} \). # 2 \( \in \mathbb{N} \)
Assume \( j \in \mathbb{N} \). # typical element of \( \mathbb{N} \)

Then \( a_j \leq 2 \implies j < i \). #
Then \( \forall j \in \mathbb{N}, a_j \leq i \implies j < i \). # introduce universal
Then \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i \). # introduce existential
Proving a Statement about a Sequence

- **A_1**: \( a_i = i^2, \ i \in \mathbb{N} \).
- **C_1**: \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i \).

**Prove that C_1 is True for A_1.**

Let \( i = 2 \). Then \( i \in \mathbb{N} \). # 2 \( \in \mathbb{N} \)
Assume \( j \in \mathbb{N} \). # typical element of \( \mathbb{N} \)

Then \( \neg (j < i) \implies \neg (a_j \leq 2) \). #
Then \( a_j \leq 2 \implies j < i \). # impl. equivalent to contrapos.
Then \( \forall j \in \mathbb{N}, a_j \leq i \implies j < i \). # introduce universal
Then \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i \). # introduce existential
Proving a Statement about a Sequence

- **A₁**: \( a_i = i^2, i \in \mathbb{N} \).
- **C₁**: \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i \).

**Prove that C₁ is True for A₁.**

Let \( i = 2 \). Then \( i \in \mathbb{N} \).  
Assume \( j \in \mathbb{N} \).  
Assume \( \neg (j < i) \).  
Then \( a_j > 2 \).  
Then \( \neg (j < i) \implies \neg (a_j \leq 2) \).  
Then \( a_j \leq 2 \implies j < i \).  
Then \( \forall j \in \mathbb{N}, a_j \leq i \implies j < i \).  
Then \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i \).
Proving a Statement about a Sequence

- **A_1**: \(a_i = i^2, \ i \in \mathbb{N}\).
- **C_1**: \(\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i\).

Prove that **C_1** is True for **A_1**.

Let \(i = 2\). Then \(i \in \mathbb{N}\). \(\#\) \(2 \in \mathbb{N}\)

Assume \(j \in \mathbb{N}\). \(\#\) typical element of \(\mathbb{N}\)

Assume \(\neg (j < i)\). \(\#\) antecedent for contrapositive

Then \(j \geq 2\). \(\#\) negation of \(j < i\) when \(i = 2\)

\[\vdots\]

Then \(a_j > 2\). \(\#\)

Then \(\neg (j < i) \implies \neg (a_j \leq 2)\). \(\#\) anteced. leads to conseq.

Then \(a_j \leq 2 \implies j < i\). \(\#\) impl. equivalent to contrapos.

Then \(\forall j \in \mathbb{N}, a_j \leq i \implies j < i\). \(\#\) introduce universal

Then \(\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i\). \(\#\) introduce existential
Proving a Statement about a Sequence

- **A₁**: \(a_i = i^2, i \in \mathbb{N}\).
- **C₁**: \(\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i\).

**Prove that C₁ is True for A₁.**

Let \(i = 2\). Then \(i \in \mathbb{N}\). ◾️ 2 ∈ \(\mathbb{N}\)

Assume \(j \in \mathbb{N}\). ◾️ typical element of \(\mathbb{N}\)

Assume \(\neg (j < i)\). ◾️ antecedent for contrapositive

Then \(j \geq 2\). ◾️ negation of \(j < i\) when \(i = 2\)

Then \(a_j = j^2 \geq 2^2 = 4\). ◾️ since \(a_j = j^2\), and \(j \geq 2\)

Then \(a_j > 2\). ◾️ since \(4 > 2\)

Then \(\neg (j < i) \implies \neg (a_j \leq 2)\). ◾️ anteced. leads to conseq.

Then \(a_j \leq 2 \implies j < i\). ◾️ impl. equivalent to contrapos.

Then \(\forall j \in \mathbb{N}, a_j \leq i \implies j < i\). ◾️ introduce universal

Then \(\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \implies j < i\). ◾️ introduce existential
Proof of Multiple Quantifiers

Structure

- Prove $\forall x \in D, \exists y \in E, P(x, y)$

  Assume $x \in D$.  # $x$ is a typical element of $D$
  Let $y = \underline{\text{[choose a particular element of the domain]}}$.  # choose a particular element of the domain
  Then $y \in E$.  # this may be obvious, otherwise prove it
  
  :  # prove $P(x, y)$
  Then $P(x, y)$.
  Then $\exists y \in E, P(x, y)$.  # introduce existential
  Then $\forall x \in D, \exists y \in E, P(x, y)$.  # introduce universal
Proof of Multiple Quantifiers

**Structure**

- Prove $\exists x \in D, \forall y \in E, P(x, y)$
  
  Let $x = \_\_\$. # choose a particular element of the domain
  Then $x \in D$. # this may be obvious, otherwise prove it
  Assume $y \in E$. # $y$ is a typical element of $E$
  
  : # prove $P(x, y)$
  Then $P(x, y)$.
  Then $\forall x \in D, P(x, y)$. # introduce universal
  Then $\exists y \in E, \forall x \in D, P(x, y)$. # introduce existential
Disproving a Statement about a Sequence

- **A_2**: \( a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N} \)

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- **C_2**: \( \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i \).

**How to disprove C_2?**

Prove \( \neg C_2 \)

- \( \neg C_2 \): \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \)
Disproving a Statement about a Sequence

\[ A_2 : a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N} \]

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\[ \neg C_2 : \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \]

Prove that \( C_2 \) is False for \( A_2 \).

Assume \( i \in \mathbb{N} \). # typical element of \( \mathbb{N} \)

Then \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). # introduction of universal
Disproving a Statement about a Sequence

- **A₂**: \(a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}\)

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- \(\neg C₂ : \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i\)

Prove that \(C₂\) is **False** for \(A₂\).

Assume \(i \in \mathbb{N}\).  
# typical element of \(\mathbb{N}\)

Let \(j = \text{___}\). Then \(j \in \mathbb{N}\).

:  

Then \(\exists j \in \mathbb{N}, j > i \land a_j \neq a_i\).  
# introduction of existential

Then \(\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i\).  
# introduction of universal
Disproving a Statement about a Sequence

- \( \mathbf{A}_2 \): \( a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N} \)

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- \( \neg \mathbf{C}_2 \): \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \)

Prove that \( \mathbf{C}_2 \) is False for \( \mathbf{A}_2 \).

Assume \( i \in \mathbb{N} \). \# typical element of \( \mathbb{N} \)

Let \( j = \) _____. Then \( j \in \mathbb{N} \).

\[ \vdots \]

Then \( j > i \land a_j \neq a_i \).

Then \( \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). \# introduction of existential

Then \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). \# introduction of universal
Disproving a Statement about a Sequence

- **A₂**: \( a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N} \)

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- **¬C₂**: \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \)

Prove that **C₂** is **False** for **A₂**.

Assume \( i \in \mathbb{N} \). \( \# \) typical element of \( \mathbb{N} \)

Let \( j = \text{____} \). Then \( j \in \mathbb{N} \).

\[ \vdots \]

Then \( j > i \).

\[ \vdots \]

Then \( a_j \neq a_i \).

Then \( j > i \land a_j \neq a_i \). \( \# \) introduction of conjunction

Then \( \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). \( \# \) introduction of existential

Then \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). \( \# \) introduction of universal
Disproving a Statement about a Sequence

- **A_2**: \( a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N} \)

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- **¬C_2**: \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \)

- **Prove that C_2 is False for A_2.**

Assume \( i \in \mathbb{N} \). # typical element of \( \mathbb{N} \)

Let \( j = i + 2 \). Then \( j \in \mathbb{N} \). # \( i, 2 \in \mathbb{N} \), and \( \mathbb{N} \) is closed under +

Then \( j > i \). # \( 2 > 0 \), so \( i + 2 > i \)

\[ \vdots \]

Then \( a_j \neq a_i \).

Then \( j > i \land a_j \neq a_i \). # introduction of conjunction

Then \( \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). # introduction of existential

Then \( \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i \). # introduction of universal
Disproving a Statement about a Sequence

- **$A_2$**: $a_0 = 0, a_1 = 0, a_i = a_{i-2} + 1, i \geq 2, i \in \mathbb{N}$

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- **$\neg C_2$**: $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i$

**Prove that $C_2$ is False for $A_2$.**

Assume $i \in \mathbb{N}$. # typical element of $\mathbb{N}$

Let $j = i + 2$. Then $j \in \mathbb{N}$. # $i, 2 \in \mathbb{N}$, and $\mathbb{N}$ is closed under $+$

Then $j > i$. # $2 > 0$, so $i + 2 > i$

Then $a_j = a_{i+2} = a_i + 1$ # since $j \geq 2$ and by Def. of $A_2$

Then $a_j \neq a_i$. # $1 > 0$, so $a_i + 1 > a_i$

Then $j > i \land a_j \neq a_i$. # introduction of conjunction

$\exists j \in \mathbb{N}, j > i \land a_j \neq a_i$. # introduction of existential

Then $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \land a_j \neq a_i$. # introduction of universal
Proof of Conjunction

Structure

- Prove $\forall x \in D, P(x) \land Q(x)$

  Assume $x \in D$.  \# $x$ is a typical element of $D$
  
  $\vdash \#$ prove $P(x)$
  Then $P(x)$.
  
  $\vdash \#$ prove $Q(x)$
  Then $Q(x)$.
  Then $P(x) \land Q(x)$.  \# introduce conjunction
  Then $\forall x \in D, P(x) \land Q(x)$.  \# introduce universal