Review

Chapter 4

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Announcements

- **Additional Review Session**: Tues, Mar 31, **2-3:45pm** in **MP203**.

- **Additional Instructor Office Hours** (Bahar):
  - Mar 30, 2-4pm, BA3201
  - Apr 01, 12-1pm, BA3201
  - Apr 08, Time and location TBA.

- **Additional TA Office Hours**:
  - Mar 30, 4-6pm, BA3201
  - Mar 31, 4-6pm, BA3201
  - More to be announced!
Proving Bounds for Functions
Review: Asymptotic Notation

Reminder: Big-Oh

- \( f \in O(g) \): \( g \) is an **upper bound** of \( f \).
  - For sufficiently large values of \( n \), \( g(n) \) multiply by a constant is always greater than \( f(n) \).

\[ \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c \cdot g(n) \]

Reminder: Limits

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \]
\[ \iff \forall \varepsilon \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon \]

\( c = L + 3 \implies c > L \)
Proving Bounds for Polynomial Expressions

Proving $O$ using Limits

Prove $f \in O(g)$

1. $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$

2. $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

3. $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does NOT exist.
Proving \( \mathcal{O} \) using Limits

- Suppose \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \).

- Prove \( f \in \mathcal{O}(g) \):
  1. Choose any value larger than \( L \) for \( c \).
  2. Assume \( f(n) \leq cg(n) \). Find a value for \( n \) such that the inequality holds.
  3. \( B \) must be larger than or equal to that value.
Proving Bounds for Polynomial Expressions

Suppose \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).

Prove \( f \in O(g) \):

1. Assume \( c = 1 \).
2. Assume \( f(n) \leq cg(n) \). Find a value for \( n \) such that the inequality holds.
3. \( B \) must be larger than or equal to that value.
Proving Bounds for Polynomial Expressions

Proving $O$ using Limits

- Suppose $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does NOT exist.

- Prove $f \in O(g)$:
  1. Find a function $h(n)$ such that $\lim_{n \to \infty} h(n)$ exists, and $\frac{f(n)}{g(n)} \leq h(n)$ for a sufficiently large value $n_1$ of $n$.
  2. Choose a value for $c$ such that $c > \lim_{n \to \infty} h(n)$.
  3. Assume $f(n) \leq cg(n)$. Find a value $n_2$ for $n$ such that the inequality holds.
  4. $B$ must be larger than or equal to $\max(n_1, n_2)$. 
Review: Asymptotic Notation

Reminder: Big-$\Omega$

- $f \in \Omega(g)$: $g$ is an **lower bound** of $f$.
  - For sufficiently large values of $n$, $g(n)$ multiply by a constant is always **less than** $f(n)$.

- $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq c.g(n)$

Reminder: Limits

- $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$
  - $\iff$
  - $\forall \varepsilon \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon$

$C = L - \varepsilon \Rightarrow C < L$
Proving Bounds for Polynomial Expressions

Proving $\Omega$ using Limits

Prove $f \in \Omega(g)$

1. $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$

2. $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

3. $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does NOT exist.
Proving Bounds for Polynomial Expressions

Proving $\Omega$ using Limits

Suppose $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$.

Prove $f \in \Omega(g)$:
1. Choose any positive value less than $L$ for $c$.
2. Assume $f(n) \geq cg(n)$. Find a value for $n$ such that the inequality holds.
3. $B$ must be larger than or equal to that value.
Proving Bounds for Polynomial Expressions

Proving $\Omega$ using Limits

Suppose $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

Prove $f \in \Omega(g)$:
1. Assume $c = 1$.
2. Assume $f(n) \geq cg(n)$. Find a value for $n$ such that the inequality holds.
3. $B$ must be larger than or equal to that value.
Proving Bounds for Polynomial Expressions

Proving \( \Omega \) using Limits

- Suppose \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \) does NOT exist.

Prove \( f \in \Omega(g) \):

1. Find a function \( h(n) \) such that \( \lim_{n \to \infty} h(n) \) exists, and \( h(n) \leq \frac{f(n)}{g(n)} \) for a sufficiently large value \( n_1 \) of \( n \).

2. Choose a value for \( c \) such that \( c < \lim_{n \to \infty} h(n) \).

3. Assume \( f(n) \geq cg(n) \). Find a value \( n_2 \) for \( n \) such that the inequality holds.

4. \( B \) must be larger than or equal to \( \max(n_1, n_2) \).
Review: Asymptotic Notation

Big-Theta

- \( f \in \Theta(g) \): \( g \) is a **tight bound** of \( f \).
  - For sufficiently large values of \( n \), \( g(n) \) is both an **upper bound** and a **lower bound** for \( f(n) \).

- \( \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1.g(n) \leq f(n) \leq c_2.g(n) \)

Proving Big-Theta

- Find \( c_1 \) and \( B_1 \) such that
  \[
  \exists c_1 \in \mathbb{R}^+, B_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_1 \Rightarrow c_1.g(n) \leq f(n)
  \]

- Find \( c_2 \) and \( B_2 \) such that
  \[
  \exists c_2 \in \mathbb{R}^+, B_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_2 \Rightarrow f(n) \leq c_2.g(n)
  \]

- Then \( B = \max(B_1, B_2) \).