CSC165 Mathematical Expression and Reasoning for Computer Science

Chapter 4: Algorithm Analysis and Asymptotic Notation

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Asymptotic notation

- $\mathcal{O}$
- $\Omega$
- $\Theta$
Here is a precise definition of “The set of functions that are eventually no more than $f$, to within a constant factor”:

**Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ (i.e., any function mapping naturals to nonnegative reals), let

$$
\mathcal{O}(f) = \{ g : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n) \}.
$$

$g \in \mathcal{O}(f)$ means that “$g$ grows no faster than $f$”. Equivalently, “$f$ is an upper bound for $g$”.

$\mathbb{R}^+$: the set of positive real numbers
Definition: For any function $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, let

$$\Omega(f) = \{ g : \mathbb{N} \to \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n) \}.$$ 

“$g \in \Omega(f)$” expresses the concept that “$g$ grows at least as fast as $f$”; $f$ is a lower bound on $g$. 
**Definition:** For any function \( f : \mathbb{N} \to \mathbb{R}_{\geq 0} \), let

\[
\Theta(f) = \{ g : \mathbb{N} \to \mathbb{R}_{\geq 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n) \}.
\]

“\( g \in \Theta(f) \)” expresses the concept that “\( g \) grows at the same rate as \( f \).”

\( f \) is a tight bound for \( g \), or \( f \) is both an upper bound and a lower bound on \( g \).
Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6$ is in $\mathcal{O}(n^2 - 4n^5 + 6n^8)$

We begin with ...

Let $c' = ____$. Then $c' \in \mathbb{R}^+$.  
Let $B' = ____$. Then $B' \in \mathbb{N}$.  
Assume $n \in \mathbb{N}$ and $n \geq B'$.  # arbitrary natural number and antecedent  
Then $2n^3 - 5n^4 + 7n^6 \leq \ldots \leq c'(n^2 - 4n^5 + 6n^8)$.  
Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow 2n^3 - 5n^4 + 7n^6 \leq c'(n^2 - 4n^5 + 6n^8)$.  # introduce $\Rightarrow$ and $\forall$  
Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 2n^3 - 5n^4 + 7n^6 \leq c(n^2 - 4n^5 + 6n^8)$.  # introduce $\exists$
Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6 \in \mathcal{O}(n^2 - 4n^5 + 6n^8)$

To fill in the . . .

we try to form a chain of inequalities, working from both ends, simplifying the expressions:

$$2n^3 - 5n^4 + 7n^6 \leq 2n^3 + 7n^6 \quad \text{(drop $-5n^4$)}$$
$$\leq 2n^6 + 7n^6 \quad \text{(increase $n^3$ to $n^6$)}$$
$$= 9n^6 \leq 9n^8 \quad \text{(simpler to compare)}$$
$$= 2(9/2)n^8 \quad \text{(choose $c' = 9/2$)}$$
$$= 2cn^8$$
$$= c'(-4n^8 + 6n^8) \quad \text{(bottom up: decrease $-4n^5$ to $-4n^8$)}$$
$$\leq c'(-4n^5 + 6n^8) \quad \text{(bottom up: drop $n^2$)}$$
$$\leq c'(n^2 - 4n^5 + 6n^8)$$

We never needed to restrict $n$ for $n \in \mathbb{N}$ ($n \geq 0$), so we can fill in $c' = 9/2$, $B' = 0$, and complete the proof.
Here are some general results that we now have the tools to prove.

- \( 3n^2 + 2n \in \mathcal{O}(n^2) \).
- \( n^3 \notin \mathcal{O}(3n^2) \).
- \( 2^n \notin \mathcal{O}(n^2) \).
- \( n^2 + n \in \Omega(15n^2 + 3) \).
Intuitively, big-Oh notation expresses something about how two functions compare as \( n \) tends toward infinity. But we know of another mathematical notion that captures a similar (though not identical) idea: the concept of \( \textit{limit} \).

Precisely, recall the following definition, for all \( L \in \mathbb{R}^{\geq 0} \):

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = L \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon
\]
Calculus

Intuitively, big-Oh notation expresses something about how two functions compare as $n$ tends toward infinity. But we know of another mathematical notion that captures a similar (though not identical) idea: the concept of limit.

Precisely, recall the following definition, for all $L \in \mathbb{R}^\geq$:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = L \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon$$

and the following special case:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \frac{f(n)}{g(n)} > \varepsilon$$
Prove: $f(n) \in \mathcal{O}(g(n))$

Suppose that $\lim_{n \to \infty} f(n)/g(n) = L$. Intuitively, this tells us that $f(n)/g(n) \approx L$, for $n$ “large enough.”

In that case, $f(n) \approx Lg(n)$ for $n$ large enough, so we should be able to prove that $f \in \mathcal{O}(g)$:

Assume $\lim_{n \to \infty} f(n)/g(n) = L$. Then $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - 0.9 < f(n)/g(n) < L + 0.9$. # definition of limit for $\varepsilon = 0.9$

Then $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) < (L + 0.9)g(n) \leq (L + 1)g(n)$.

Then $f \in \mathcal{O}(g)$. # definition of $\mathcal{O}$, with $B = n_0$ and $c = L + 1$

Hence, $\lim_{n \to \infty} f(n)/g(n) = L \Rightarrow f \in \mathcal{O}(g)$. 

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Prove: \( g(n) \not\in \mathcal{O}(f(n)) \)

Recall that \( g(n) = 2^n \) and \( f(n) = n \). We rely on the fact that \( \lim_{n \to \infty} \frac{2^n}{n} = \infty \).\(^1\)

Assume \( c \in \mathbb{R}^+ \), assume \( B \in \mathbb{N} \). \# arbitrary values

Then \( \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 2^n / n > c \). \# definition of \( \lim_{n \to \infty} \frac{2^n}{n} = \infty \) with \( \varepsilon = c \)

Let \( n_0 \) be such that \( \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 2^n / n > c \), and \( n' = \max(B, n_0) \).

Then \( n' \in \mathbb{N} \).

Then \( n' \geq B \). \# by definition of \( \max \)

Then \( 2^{n'} > cn' \) because \( 2^{n'} / n' > c \). \# by the first line above, since \( n' \geq n_0 \)

Then \( n' \geq B \land g(n') \geq cf(n') \). \# introduce \( \land \)

Then \( \exists n \in \mathbb{N}, n \geq B \land g(n) \geq cf(n) \). \# introduce \( \exists \)

Then \( \forall c \in \mathbb{R}, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land g(n) > cf(n) \). \# introduce \( \forall \)

\(^1\) Applying l'Hôpital's Rule, \( \lim_{n \to \infty} \frac{2^n}{n} = \lim_{n \to \infty} \frac{\ln(2) \cdot 2^n}{1} = \infty \).
Here are some general results that we now have the tools to prove.

- $3n^2 + 2n \in O(n^2)$.
- $n^3 \notin O(3n^2)$.
- $2^n \notin O(n^2)$.
- $n^2 + n \in \Omega(15n^2 + 3)$. 
Here are some general results that we now have the tools to prove.

- \((f \in \mathcal{O}(g) \land g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)\).
  Intuition: If \(f\) grows no faster than \(g\), and \(g\) grows no faster than \(h\), then \(f\) must grow no faster than \(h\).

- \(g \in \Omega(f) \iff f \in \mathcal{O}(g)\).
  Intuition: if \(f\) grows no faster than \(g\), then \(g\) grows no slower than \(f\).

- \(g \in \Theta(f) \iff g \in \mathcal{O}(f) \land g \in \Omega(f)\).
  Intuition: \(g\) grows at the same rate as \(f\). \(f\) is both an upper bound and a lower bound on \(g\).
Theorem 1

For any functions $f, g, h : \mathbb{N} \to \mathbb{R}_{\geq 0}$, we have $(f \in O(g) \land g \in O(h)) \Rightarrow f \in O(h)$. 

**Proof:**

Assume $f \in O(g) \land g \in O(h)$.

So $f \in O(g)$.

So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n > B \Rightarrow f(n) \leq cg(n)$.  # by def’n of $f \in O(g)$

Let $c_g \in \mathbb{R}^+, B_g \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_g \Rightarrow f(n) \leq c_g g(n)$.

So $g \in O(h)$.

So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq ch(n)$.  # by def’n of $g \in O(h)$

Let $c_h \in \mathbb{R}^+, B_h \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_h \Rightarrow g(n) \leq c_h h(n)$.

Let $c' = c_g c_h$. Let $B' = \max(B_g, B_h)$.

Then, $c' \in \mathbb{R}^+$ (because $c_g, c_h \in \mathbb{R}^+$) and $B' \in \mathbb{N}$ (because $B_g, B_h \in \mathbb{N}$).

Assume $n \in \mathbb{N}$ and $n \geq B'$.

Then $n \geq B_h$ (by definition of $\max$), so $g(n) \leq c_h h(n)$.

Then $n \geq B_g$ (by definition of $\max$), so $f(n) \leq c_g g(n) \leq c_g c_h h(n)$.

So $f(n) \leq c'h(n)$.

Hence, $\forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) \leq c'h(n)$.

Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq ch(n)$.

So $f \in O(g)$, by definition.
Theorem 2

For any functions \( f, g : \mathbb{N} \to \mathbb{R}_{\geq 0} \), we have \( g \in \Omega(f) \iff f \in O(g) \).

**Proof:**

\[
g \in \Omega(f) \\
\iff \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n) \text{ (by definition)}
\]
\[
\iff \exists c' \in \mathbb{R}^+, \exists B' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) \leq c'g(n)
\]

(letting \( c' = 1/c \) and \( B' = B \))
\[
\iff f \in O(g) \text{ (by definition)}
\]
Theorem 3

For any functions $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, we have $g \in \Theta(f) \iff g \in \mathcal{O}(f) \land g \in \Omega(f)$.

**Proof:**

$g \in \Theta(f)$

$\iff$ (by definition)

$\exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n)$.

$\iff$ (combined inequality, and $B = \max(B_1, B_2)$)

$\left( \exists c_1 \in \mathbb{R}^+, \exists B_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_1 \Rightarrow g(n) \geq c_1 f(n) \right) \land$

$\left( \exists c_2 \in \mathbb{R}^+, \exists B_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_2 \Rightarrow g(n) \leq c_2 f(n) \right)$

$\iff$ (by definition)

$g \in \Omega(f) \land g \in \mathcal{O}(f)$
Corollary: For any functions $f, g : \mathbb{N} \to \mathbb{R}^{>0}$, we have $g \in \Theta(f) \iff f \in \Theta(g)$.

Proof:

\[
g \in \Theta(f) \\
\iff g \in \mathcal{O}(f) \land g \in \Omega(f) \quad \text{(by 3)} \\
\iff g \in \mathcal{O}(f) \land f \in \mathcal{O}(g) \quad \text{(by 2)} \\
\iff f \in \mathcal{O}(g) \land g \in \mathcal{O}(f) \quad \text{(by commutativity of \(\land\))} \\
\iff f \in \mathcal{O}(g) \land f \in \Omega(g) \quad \text{(by 2)} \\
\iff f \in \Theta(g) \quad \text{(by 3)}
\]