Asymptotic notation

- $O$
- $\Omega$
- $\Theta$
Here is a precise definition of “The set of functions that are eventually no more than $f$, to within a constant factor”:

**Definition:** For any function $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ (i.e., any function mapping naturals to nonnegative reals), let

$$\mathcal{O}(f) = \{ g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} | \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n) \}.$$ 

$g \in \mathcal{O}(f)$ means that “$g$ grows no faster than $f$”. Equivalently, “$f$ is an upper bound for $g$”.

$\mathbb{R}^+$: the set of positive real numbers
Proof Example

Suppose: \( g(n) = 3n^2 + 2 \) and \( f(n) = n^2 \)

Then \( g \in O(f) \).

To be more precise, we need to prove the statement

\[
\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2 \leq cn^2.
\]

Find some \( c \) and \( B \) that “work” in order to prove the theorem.
Prove: \( \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2 \leq cn^2 \)

Idea:
Finding \( c \) means finding a factor that will scale \( n^2 \) up to the size of \( 3n^2 + 2 \). Setting \( c = 3 \) almost works, but there’s that annoying additional term 2. Certainly \( 3n^2 + 2 < 4n^2 \) so long as \( n \geq 2 \), since \( n \geq 2 \Rightarrow n^2 > 2 \). So pick \( c = 4 \) and \( B = 2 \)
Prove: $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2 \leq cn^2$

Idea:

Finding $c$ means finding a factor that will scale $n^2$ up to the size of $3n^2 + 2$. Setting $c = 3$ almost works, but there’s that annoying additional term 2. Certainly $3n^2 + 2 < 4n^2$ so long as $n \geq 2$, since $n \geq 2 \Rightarrow n^2 > 2$. So pick $c = 4$ and $B = 2$

Let $c' = 4$ and $B' = 2$.
Then $c' \in \mathbb{R}^+$ and $B' \in \mathbb{N}$.
Assume $n \in \mathbb{N}$ and $n \geq B'$. # direct proof for an arbitrary natural number
Then $n^2 \geq B'^2 = 4$. # squaring is monotonic on natural numbers
Then $n^2 \geq 2$.
Then $3n^2 + n^2 \geq 3n^2 + 2$. # adding $3n^2$ to both sides of the inequality
Then $3n^2 + 2 \leq 4n^2 = c'n^2$ # re-write
Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow 3n^2 + 2 \leq c'n^2$ # introduce $\forall$ and $\Rightarrow$
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 3n^2 + 2 \leq cn^2$. # introduce $\exists$ (twice)

So, by definition, $g \in \mathcal{O}(f)$. 
Ω, Θ Notation

By analogy with $\mathcal{O}(f)$, consider two other definitions:

Definition: For any function $f : \mathbb{N} \to \mathbb{R}^\geq 0$, let

$$\Omega(f) = \{ g : \mathbb{N} \to \mathbb{R}^\geq 0 \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n) \}.$$  

“$g \in \Omega(f)$” expresses the concept that “$g$ grows at least as fast as $f$”; $f$ is a lower bound on $g$. 
Ω, Θ Notation

By analogy with $O(f)$, consider two other definitions:

**Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^\geq 0$, let

$$\Omega(f) = \{ g : \mathbb{N} \to \mathbb{R}^\geq 0 \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n) \}.$$

“$g \in \Omega(f)$” expresses the concept that “$g$ grows at least as fast as $f$”; $f$ is a lower bound on $g$.

**Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^\geq 0$, let

$$\Theta(f) = \{ g : \mathbb{N} \to \mathbb{R}^\geq 0 \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n) \}.$$

“$g \in \Theta(f)$” expresses the concept that “$g$ grows at the same rate as $f$”. $f$ is a tight bound for $g$, or $f$ is both an upper bound and a lower bound on $g$. 
Next week: more complex examples
Another $O$ Proof

Prove that $2n^3 - 5n^4 + 7n^6$ is in $O(n^2 - 4n^5 + 6n^8)$

We begin with ...

Let $c' = \text{____}$. Then $c' \in \mathbb{R}^+$. 
Let $B' = \text{____}$. Then $B' \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ and $n \geq B'$. # arbitrary natural number and antecedent

Then $2n^3 - 5n^4 + 7n^6 \leq \ldots \leq c'(n^2 - 4n^5 + 6n^8)$.

Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow 2n^3 - 5n^4 + 7n^6 \leq c'(n^2 - 4n^5 + 6n^8)$. # introduce $\Rightarrow$

and $\forall$

Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 2n^3 - 5n^4 + 7n^6 \leq c(n^2 - 4n^5 + 6n^8)$. # introduce $\exists$
Another $\mathcal{O}$ Proof

Prove that $2n^3 - 5n^4 + 7n^6 \in \mathcal{O}(n^2 - 4n^5 + 6n^8)$

To fill in the . . .

we try to form a chain of inequalities, working from both ends, simplifying the expressions:

\begin{align*}
2n^3 - 5n^4 + 7n^6 &\leq 2n^3 + 7n^6 \quad \text{(drop $-5n^4$)} \\
&\leq 2n^6 + 7n^6 \quad \text{(increase $n^3$ to $n^6$)} \\
&= 9n^6 \leq 9n^8 \quad \text{(simpler to compare)} \\
&= 2(9/2)n^8 \quad \text{(choose $c' = 9/2$)} \\
&= 2cn^8 \\
&= c'(-4n^8 + 6n^8) \quad \text{(bottom up: decrease $-4n^5$ to $-4n^8$)} \\
&\leq c'(-4n^5 + 6n^8) \quad \text{(bottom up: drop $n^2$)} \\
&\leq c'(n^2 - 4n^5 + 6n^8)
\end{align*}

We never needed to restrict $n$ for $n \in \mathbb{N} \ (n \geq 0)$, so we can fill in $c' = 9/2$, $B' = 0$, and complete the proof.