CSC165, Winter 2015
Assignment 2
Sample Solutions

IMPORTANT: You must use the proof structures and format of this course. Otherwise, you won’t get full mark even if your answers are correct.

1. Prove or disprove each of the following claims.

(a) \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil. \)

Solution: The claim is false. I will disprove it by proving the negation of the claim which is the following statement:

\[ \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil. \]

proof:

Let \( x = 1.2, y = 1.2. \) then \( x, y \in \mathbb{R}. \) \# since \( 1.2 \in \mathbb{R} \)
then \( \lceil x + y \rceil = 3. \) \# by definition of the ceiling function since \( x + y = 2.4 \)
then \( \lceil x \rceil + \lceil y \rceil = 4. \) \# by definition of the ceiling function since \( \lceil x \rceil = \lceil y \rceil = 2 \)
then \( \lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil. \) \# \( 3 \neq 4 \)
then \( \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil. \) \# introduced \( \exists \)
(b) For all integers \( x, y, \) and \( z \), if \( x \nmid y.z \) then \( x \nmid y \) and \( x \nmid z \). (Note that the symbol \( \nmid \) denotes “does not divide”)

**Solution:** The claim is true. Here’s the translation of the claim in logical form:

\[
\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, (x \nmid y.z) \Rightarrow (x \nmid y) \land (x \nmid z).
\]

**Proof:** # prove by contrapositive

Assume \( x, y, z \in \mathbb{Z} \). # \( x, y, z \) are typical integers

assume \((x \mid y) \lor (x \mid z)\) # antecedent of contrapositive

Case 1: Assume \( x \mid y \) # antecedent

then \( \exists k_1 \in \mathbb{Z} \) such that \( y = k_1 \times x \) # definition of | 

then \( y \times z = k_1 \times x \times z \) # multiple both sides by \( z \)

then \( \exists k \in \mathbb{Z} \) such that \( y \times z = k \times x \) # let \( : k = k_1 \times z, k \in \mathbb{Z}, \) closed under \( \times \)

then \( x \mid y \times z \) # definition of | 

Case 2: Assume \( x \mid z \) # antecedent

then \( \exists k_2 \in \mathbb{Z} \) such that \( z = k_2 \times x \) # definition of | 

then \( y \times z = k_2 \times x \times z \) # multiple both sides by \( y \)

then \( \exists k \in \mathbb{Z} \) such that \( y \times z = k \times x \) # let \( : k = k_2 \times y, k \in \mathbb{Z}, \) closed under \( \times \)

then \( x \mid y \times z \) # definition of | 

then \( \exists k \in \mathbb{Z} \) such that \( y \times z = k \times x \) # let \( : k = k_1 \times z, k \in \mathbb{Z}, \) closed under \( \times \)

then \( x \mid y \times z \) # true for both cases

then \((x \mid y) \lor (x \mid z) \Rightarrow (x \mid y \times z)\) # introduced \( \Rightarrow \)

then \((x \nmid y \times z) \Rightarrow (x \mid y) \land (x \mid z)\) # implication is equivalent to its contrapositive

then \( \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}, (x \nmid y \times z) \Rightarrow (x \nmid y) \land (x \nmid z) \) # introduced \( \forall \)
2. Use proof by contradiction to prove that for all prime numbers \(x, y\), and \(z\), \(x^2 + y^2 \neq z^2\).

**Solution:** Here’s the translation of the claim in logical form:

Let \(P\): set of all prime numbers, prove SP:

\[ \forall x \in P, \forall y \in P, \forall z \in P, (x^2 + y^2 \neq z^2). \]

Negation of SP:

\[ \exists x \in P, \exists y \in P, \exists z \in P, (x^2 + y^2 = z^2). \]

**Proof:** # prove by contradiction

Assume \( \exists x \in P, \exists y \in P, \exists z \in P, (x^2 + y^2 = z^2) \). # derive contradiction \( \neg \text{SP} \)

Let \( x_0, y_0, z_0 \in P \) such that \( x_0^2 + y_0^2 = z_0^2 \) # instantiate \( \exists \)
then \( x_0^2 = z_0^2 - y_0^2 = (z_0 - y_0)(z_0 + y_0) \). # algebra
then factors of \( x^2 \) are 1, \( x, x^2 \). # \( x \) is a prime number
and \( (z_0 + y_0) \neq (z_0 - y_0) \). # \( y_0 \) and \( z_0 \) are primes, \( y_0 \neq 0 \)
then \( ((z_0 - y_0 = 1) \land (z_0 + y_0 = x_0^2)) \lor ((z_0 + y_0 = 1) \land (z_0 - y_0 = x_0^2)) \)
# \((z_0 - y_0) \) and \((z_0 + y_0)\) can only be 1 or \(x_0^2\), as both are interger factors.
Case 1: Assume \((z_0 - y_0 = 1) \land (z_0 + y_0 = x_0^2)\).
then \( z_0 \) is the successor of \( y_0 \). # \( z_0 - y_0 = 1 \)
then \( z_0 = 3 \) and \( y_0 = 2 \). # 2 and 3 are the only successive primes, \( P = \{2, 3, 5, \ldots\} \)
then \( z_0 + y_0 = 5 \). # \( z_0 = 3 \) and \( y_0 = 2 \)
Contradiction! # \( z_0 + y_0 = x_0^2 = 5 \) but 5 is not square of any interger, contradict to \( x_0 \in P \)

Case 2: Assume \((z_0 + y_0 = 1) \land (z_0 - y_0 = x_0^2)\).
then \( y_0 + z_0 \geq 4 \). # \( y_0 \) and \( z_0 \) are primes, and so \( y_0 \geq 2, z_0 \geq 2 \)
Contradiction! # \( y_0 + z_0 \geq 4 \) contradict to \( z_0 + y_0 = 1 \)

Contradiction! # contradiction in both cases
then \( \forall x \in P, \forall y \in P, \forall z \in P, (x^2 + y^2 \neq z^2) \) # negation of the statement \((\neg \text{SP})\) is false, then SP
3. Consider the definition:

\[
\text{Def}_1 : \forall x \in \mathbb{R}, \forall y \in \mathbb{Z}, (y = \lfloor x \rfloor) \iff (y \leq x) \land (\forall z \in \mathbb{Z}, (z \leq x) \Rightarrow (z \leq y)).
\]

Use the proof structures of this course and \text{Def}_1 to prove the following claims

(a) \( S_1 : \forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \leq y) \land (y < 1) \Rightarrow ([n + y] = n). \)

Note: In your proof, you may ONLY use those properties of the floor function that are specified by \text{Def}_1.

Proof: \# direct proof

Assume \( n \in \mathbb{Z}, y \in \mathbb{R} \) \# \( y \) is a typical real number, \( n \) is a typical integer
assume \((0 \leq y) \land (y < 1)\). \# antecedent
assume \( z \in \mathbb{Z} \). \# \( z \) is a typical integer
assume \( z \leq n + y \). \# antecedent
then \((n \leq n + y)\) \# \( 0 \leq y \), add \( n \) to both sides of the inequalities
then \((n + y < n + 1)\) \# \( y < 1 \), add \( n \) to both sides of the inequalities
\( z < n + 1 \). \# \( n + y < n + 1 \) and transitivity of \(< \)
then \( z \leq n \) \# \( n < n + 1 \) and \( z \in \mathbb{Z} \), and there is no integer between two successor integers
then \( [n \leq n + y] \Rightarrow (z \leq n). \) \# introduced \( \Rightarrow \)
then \( \forall z \in \mathbb{Z}, (z \leq n + y) \Rightarrow (z \leq n). \) \# introduced \( \forall \)
then \((n \leq n + y) \land (\forall z \in \mathbb{Z}, (z \leq n + y) \Rightarrow (z \leq n)). \) \# introduced \( \land \)
then \([n + y] = n. \) \# by \text{Def}_1
then \((0 \leq y) \land (y < 1) \Rightarrow ([n + y] = n). \) \# introduced \( \Rightarrow \)
then \( \forall n \in \mathbb{Z}, \forall y \in \mathbb{R}, (0 \leq y) \land (y < 1) \Rightarrow ([n + y] = n). \) \# introduced \( \forall \)