1. Let $B(x)$ denotes “$x$ plays this 2-person board game” and $P(x, y)$ denotes “$x$ has played this 2-person board with $y$”, where the domain for variables $x$ and $y$ consists of all students in your class.

Questions:

- Express each of the following sentences in logical notation. Define all sets and predicate symbols that you use in the logical expressions.
- Write the negation of each of the sentences in English and in logical form. Simplifying the logical sentences so that only predicates are negated.

Solution: Let $C$ denotes the set of all students in your class.

(a) [4 Marks] If Carol and Dan play this 2-person board game, then they have played it with each other.

(Hint: note that $P(x, y)$ only expresses that $x$ has played with $y$; this does not necessarily mean that $y$ has played with $x$).

$$B(Carol) \land B(Dan) \Rightarrow P(Carol, Dan) \land P(Dan, Carol).$$

Negation: Carol and Dan play this 2-person board game, but they have not played it with each other.

$$B(Carol) \land B(Dan) \land (\neg P(Carol, Dan) \lor \neg P(Dan, Carol)).$$

(b) [3 Marks] No one in the class has played this 2-person board game with Bob.

$$\forall x \in C, \neg P(x, Bob).$$

Negation: Someone in the class has played this 2-person board game with Bob.

$$\exists x \in C, P(x, Bob).$$

(c) [3 Marks] Someone in your class does not play this 2-person board game.

$$\exists x \in C, \neg B(x).$$

Negation: All people in your class play this 2-person board game.

$$\forall x \in C, B(x).$$
(d) **[5 Marks]** Everyone in your class who plays this 2-person board game has played the game with at least one other student in your class.

\[ \forall x \in C, B(x) \Rightarrow (\exists y \in C, (x \neq y) \land P(x, y)) \]

**Negation:** Someone in your class plays this 2-person board game, but has not played it with other student in the class.

\[ \exists x \in C, B(x) \land (\forall y \in C, (x = y) \lor \neg P(x, y)) \]

(e) **[4 Marks]** A student in your class has played this 2-person board game with everyone in your class.

\[ \exists x \in C, \forall y \in C, P(x, y) \]

**Negation:** No one in your class has played this 2-person board game with everyone in the class.

\[ \forall x \in C, \exists y \in C, \neg P(x, y) \]

2. Let \( S \) denotes the set of all students in your class, \( C \) denotes the set of all countries, and \( T(x, y) \) denotes that \( x \) travels to country \( y \).

Express each of the following statements by a simple English sentence.
Avoid symbols (e.g. \( x \)) and predicates (e.g. \( T(x,y) \)) in English sentences.

(a) **[1 Mark]** \( \neg T(Bob, y) \).

**Solution:** Bob does not travel to the country.

(b) **[1 Mark]** \( \exists x \in S, T(x, Denmark) \land \forall x \in S, T(x, France) \).

**Solution:** Some students in your class travel to Denmark and all students in the class travel to France.

(c) **[1 Mark]** \( \exists y \in C, T(Mona, y) \lor T(Pete, y) \).

**Solution:** At least one of Mona or Pete travels to some country.

(d) **[2 Marks]** \( \forall x \in S, \forall z \in S, \exists y \in C, (x \neq z) \Rightarrow \neg(T(x, y) \land T(z, y)) \).

**Solution:** For every two different students in your class, exists a country such that at most one of the two students travel to.

(e) **[2 Marks]** \( \exists x \in S, \exists z \in S, \forall y \in C, (x \neq z) \land (T(x, y) \Leftrightarrow T(z, y)) \).

**Solution:** At least two different students in your class travel to the exact same countries.

(f) **[2 Marks]** \( \forall x \in S, \forall z \in S, \exists y \in C, (T(x, y) \Leftrightarrow T(z, y)) \).

**Solution:** For every two students in your class, there exists a country that either they both travel to, or none of them travel to.

3. Give the contrapositive and converse of the following statements in English.
(Hint: it might be helpful if you first write these sentences in logical notation)

(a) **[2 Marks]** If Socrates is a man and all men are mortal, then Socrates is mortal.

**Contrapositive:** If Socrates is not mortal, then he is not a man or exists a man who is not mortal.

**Converse:** If Socrates is mortal, then he is a man and all men are mortal.
(b) [2 Marks] If it does not rain or it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on.

Contrapositive: If the sailing race won’t be held or the lifesaving demonstration won’t go on, then it rains and it is foggy.

Converse: If the sailing race will be held and the lifesaving demonstration will go on, then it does not rain or it is not foggy.

(c) [2 Marks] For me, playing hockey is sufficient for being sore the next day.

Contrapositive: If I am not sore the next day, then I didn’t play hockey.

Converse: If I am sore the next day, then I played hockey.

4. Suppose T is a set of natural numbers, and we have the following statement about T.

S\(_1\): 2 is the only prime number that divides elements of T.

Which of the following statements imply S\(_1\)? Which are implied by S\(_1\)? For each case, give a brief justification.

(a) [6 Marks] S\(_2\): T is the set of all natural numbers that satisfy the quadratic equation \(x^2 + x + 1 = 0\).

S\(_2\) implies S\(_1\) because S\(_2\) is an empty set because the implication is vacuously true. In other words, all elements in the empty set satisfy S\(_1\).

S\(_1\) does not imply S\(_2\) because there are non-empty sets that satisfy S\(_1\).

(b) [6 Marks] S\(_3\): \(T = \{16, 8, 528\}\).

S\(_1\) and S\(_3\) do not imply each other: 528 is dividable by 3 and 11, so whenever S\(_1\) is True S\(_3\) is False and whenever S\(_3\) is True S\(_1\) is False.

(c) [6 Marks] S\(_4\): All elements of T are equal to \(2^n\), where n is a natural number.

S\(_1\) implies S\(_4\). If the only prime factor of elements in T is 2, then every element of T is a power of 2. However, \(2^0 = 1\) is not dividable by 2, so S\(_4\) does not imply S\(_1\).

(d) [6 Marks] S\(_5\): If \(i, j \in T\), and \(i < j\), then \(i\) divides \(j\).

S\(_1\) implies S\(_5\), since if S\(_1\) is true, then smaller elements of T divide larger elements. However, S\(_5\) doesn’t imply S\(_1\), since \(\{3, 6, 12\}\) satisfies S\(_5\) but not S\(_1\).

5. Prove that the following equivalences are tautology. Show your work in detail. You may not use truth tables or Venn diagrams in your proof, but you are allowed to use manipulation rules (presented in Section 2.17 of the course notes).

For each step, you must clearly state which rule you apply.

(a) [6 Marks] \(((P \Rightarrow Q_1) \land (P \Rightarrow Q_2)) \Leftrightarrow (P \Rightarrow (Q_1 \land Q_2))\)

Solution:

\[
(P \Rightarrow (Q_1 \land Q_2)) \iff (\neg P \lor (Q_1 \land Q_2)) \quad \text{implication}
\]

\[
\iff ((\neg P \lor Q_1) \land (\neg P \lor Q_2)) \quad \text{distribute } \lor \text{ over } \land
\]

\[
\iff ((P \Rightarrow Q_1) \land (P \Rightarrow Q_2)) \quad \text{implication}
\]

(b) [16 Marks] \(((\exists x \in D, P(x) \lor Q(x)) \Rightarrow R(y)) \Leftrightarrow (((\exists x \in D, P(x)) \Rightarrow R(y)) \land ((\exists x \in D, Q(x)) \Rightarrow R(y)))\)
Solution:

\[ (((\exists x \in D, P(x) \lor Q(x)) \Rightarrow R(y)) \iff \neg (\exists x \in D, P(x) \lor Q(x)) \lor R(y)) \]  
implication

\[ \iff ((\forall x \in D, \neg (P(x) \lor Q(x))) \lor R(y)) \]  
quantifier negation

\[ \iff ((\forall x \in D, \neg P(x) \land \neg Q(x)) \lor R(y)) \]  
De Morgan’s

\[ \iff (((\forall x \in D, \neg P(x)) \land (\forall x \in D, \neg Q(x))) \lor R(y)) \]  
quantifier distributive

\[ \iff (((\forall x \in D, \neg P(x)) \lor R(y)) \land ((\forall x \in D, \neg Q(x)) \lor R(y))) \]  
distribute \lor over \land

\[ \iff (\neg (\forall x \in D, \neg P(x)) \Rightarrow R(y)) \land (\neg (\forall x \in D, \neg Q(x)) \Rightarrow R(y)) \]  
implication

\[ \iff (((\exists x \in D, P(x)) \Rightarrow R(y)) \land ((\exists x \in D, Q(x)) \Rightarrow R(y))) \]  
quantifier negation